

# VIII

## SINGULARITIES OF FUNCTIONS REPRESENTED BY DIRICHLET SERIES WITH FINITE UPPER DENSITY

The theory of detection of singularities of a function represented by a Dirichlet series has been very much enriched in the last twenty years. The subject is, however, masterfully treated in the book of V. Bernstein [3] which we have mentioned many times in this work, and we shall not attempt to give another systematic exposé of this matter. We shall therefore not give here the striking results of Pólya and V. Bernstein which are of a high degree of precision and generality (see [3]). Let us only recall that their theory is chiefly based on the notion of maximum density  $\mathcal{D}$ , due to Pólya and mentioned (in connection with Taylor series) above. Only results where the upper density is involved will be given in this chapter.

We shall now suppose that  $-\infty < \sigma_c < \infty$ . A *curvilinear channel connected with the half-plane  $\sigma > \sigma_c$*  is a channel (as defined in Chapter IV) of which the central line contains at least one point situated in the half-plane  $\sigma > \sigma_c$ . To say that the function  $F(s)$ , represented by  $\sum a_n e^{-\lambda_n s}$ , admits in such a channel  $\Sigma$  a singularity, means that there is no function holomorphic in this channel and which, for the points of  $\Sigma$  for which  $\sigma > \sigma_c$ , is given by the sum of the Dirichlet series.

The sequences  $\{\lambda_n\}$  and  $\{a_n\}$  given, we shall set

$$\sigma(\{a_n\}, \{\lambda_n\}) = \limsup_{n \rightarrow \infty} \frac{\log |a_n| - \log \Lambda_n}{\lambda_n},$$

where  $\{\Lambda_n\}$  is the sequence associated with  $\{\lambda_n\}$ .

The following theorem can now be proved:

THEOREM XXVIII. Suppose that  $F(s)$  is represented by  $\sum a_n e^{-\lambda_n s}$  with  $\sigma_c^F < \infty$ ,  $\sigma(\{a_n\}, \{\lambda_n\}) > -\infty$ .  $F(s)$  admits a singularity in every curvilinear channel of width  $2\pi a$ , with  $a > D$ , connected with the half-plane  $\sigma > \sigma_c^F$ , of which the central line contains at least one point  $s_1 = \sigma_1 + it_1$  with

$$\sigma_1 < \sigma(\{a_n\}, \{\lambda_n\}).$$

Let  $\Sigma = \Sigma(s(u), \pi a)$  be a channel, connected with the half-plane  $\sigma > \sigma_c^F$ , of width  $2\pi a$ , the central line  $L$  of which contains a point  $s_1 = s(u_1)$  with  $\sigma_1 < \sigma(\{a_n\}, \{\lambda_n\})$ , ( $s_1 = \sigma_1 + it_1$ ). If  $F(s)$  did not have a singularity in  $\Sigma$ ,  $F(s)$  would be holomorphic and bounded in the channel  $\Sigma(s(u), \pi a_1)$ , where  $a_1$  is such that  $D < a_1 < a$ . It would then follow from Theorem XVI that

$$\sigma_1 \geq \limsup_{n \rightarrow \infty} \frac{\log |a_n| - \log \Lambda_n}{\lambda_n} = \sigma(\{a_n\}, \{\lambda_n\}),$$

contrary to the supposition that  $\sigma_1 < \sigma(\{a_n\}, \{\lambda_n\})$ .

Let us now set

$$\sigma\{\lambda_n\} = \limsup_{n \rightarrow \infty} \frac{\log \Lambda_n}{\lambda_n}.$$

It is obvious that:

$$\begin{aligned} \sigma(\{a_n\}, \{\lambda_n\}) &= \limsup_{n \rightarrow \infty} \frac{\log |a_n| - \log \Lambda_n}{\lambda_n} \\ &\geq \limsup_{n \rightarrow \infty} \frac{\log |a_n|}{\lambda_n} - \limsup_{n \rightarrow \infty} \frac{\log \Lambda_n}{\lambda_n} \\ &= \sigma_c - \sigma\{\lambda_n\}. \end{aligned}$$

It follows, on the other hand, from Theorem XVII that if  $D < \infty$  and if (79) is satisfied, then

$$\sigma\{\lambda_n\} \leq B(D, h),$$

with  $B(D, h) = 3D(3 - \log(hD))$ , if  $D > 0$ , and  $B(0, h) = 0$ .

The following theorem is therefore an immediate consequence of Theorem XXVIII.

THEOREM XXIX. If  $F(s)$  is represented by  $\sum a_n e^{-\lambda_n s}$  with  $\sigma_c^F > -\infty$  and if

$$\liminf_{n \rightarrow \infty} (\lambda_{n+1} - \lambda_n) = h > 0,$$

then there exists a singularity of  $F(s)$  in every curvilinear channel of width  $2\pi a$  with  $a > D$ , connected with the half-plane  $\sigma > \sigma_c^F$ , the central line of which contains at least one point  $s_1$  such that  $\sigma_1 < \sigma_c^F - B(D, h)$ .

This theorem contains the following theorem due to A. Ostrowski [16].

THEOREM XXX. There exists a positive function

$$\alpha(h, D) (0 < h < \infty, 0 < D < \infty)$$

tending, when  $h$  is fixed, to zero as  $D$  tends to zero, and such that, if  $F(s)$  is represented by  $\sum a_n e^{-\lambda_n s}$  with  $-\infty < \sigma_c$ , the sequence  $\{\lambda_n\}$  satisfying the condition

$$\liminf_{n \rightarrow \infty} (\lambda_{n+1} - \lambda_n) = h > 0,$$

and  $D$  being the upper density of  $\{\lambda_n\}$ , then  $F(s)$  has at least one singularity in each circle of which the center is an arbitrary point of the axis of convergence and of which the radius is  $\alpha(h, D)$ .

More precisely  $\alpha(h, D) = D[\pi + 3(3 - \log(hD))]$ .

Since  $\alpha(h, D)$  tends to zero with  $D$  ( $h$  being fixed), we have in particular the following theorem.

THEOREM XXXI. If  $F(s)$  is represented by  $\sum a_n e^{-\lambda_n s}$  where  $\sigma_c^F > -\infty$ , and where the sequence  $\{\lambda_n\}$  is such that

$$\lim_{n \rightarrow \infty} \frac{n}{\lambda_n} = 0,^1$$

$$\liminf_{n \rightarrow \infty} (\lambda_{n+1} - \lambda_n) > 0,$$

then each point of the axis of convergence is a singularity for  $F(s)$ .

<sup>1</sup>This condition is obviously equivalent to  $\lim_{x \rightarrow \infty} \frac{N(x)}{x} = 0$ . Generally the two conditions  $\limsup_{n \rightarrow \infty} \frac{n}{\lambda_n} = D$ ,  $\limsup_{x \rightarrow \infty} \frac{N(x)}{x} = D$  are equivalent.

This theorem, when the  $\lambda_n$  are integers, is due to Fabry; the general case is due to Pólya [17].

Coming back to what we have said in Chapter VII about the analogy between the two kinds of theorems: those concerning the singularities of a Taylor series (or a Dirichlet series) and those concerning lines  $J$  (or lines  $\bar{J}$ , for Dirichlet series), we shall add the following remark which is a corollary of all the results established in both chapters VII and this one:

To theorems on lines  $\bar{J}$  in a strip of width  $2\pi a$  correspond theorems on singularities in such a strip and not theorems on singularities on a segment of the axis of convergence of length  $2\pi a$ .